CSE 150 Fall 2009: Homework #5

Chao Xu

Problem 1

1. There are F_n ways, where F_n is the nth Fibonacci number. The new covering is the covering for $n-2$ cases padded with two horizontal dominoes, and $n-1$ cases padded with 1 vertical dominoes.

2. a_n is the string of n H and T with no two adjacent H's. $a_1 = 2, a_2 = 3$, then for a_n, if it end in tails, then the previous string is any of the a_{n-1} strings. If it end in heads, then the one before must be tails, thus it must be one of the a_{n-2} valid strings. $a_n = a_{n-1} + a_{n-2}$, $a_n = F_{n+1}$ for $n > 2$.

3. Let C_n be the number of ways to cover $1 \times n$ checkerboard, then we have $C_n = C_{n-1} + C_{n-3}$. Since we generate all combination by adding a 1 block on C_{n-1} and a 3 block on on C_{n-3}.

Problem 2

$\log x_n = \log(x_{n-1}^2 / x_{n-2})$.
Let $b_n = \log_2(x_n)$, then we have
$b_n = 2b_{n-1} - b_{n-2}$ with $b_1 = 0, b_2 = 1$. Solving the characteristic polynomial $(r-1)^2 = 0$
$b_1 = A + 1B$
$b_2 = A + 2B$
$A = -1$
$B = 1$
$b_n = n - 1$
thus $x_n = 2^{n-1}$

Problem 3

1. $a + b = c + d$. There are $\binom{21}{2} = 210$ possible pairs for (a, c). The possible sum range from 1 to 200. According to the pigeonhole principle, there must exist a, b, c, d such that $a + b = c + d$.

2. Only way for $(a_1 - 1)(a_2 - 2) \ldots (a_n - n)$ to be not even is for $a_1 - 1, a_3 - 3, \ldots, a_n - n$ to be odd, thus $a_1, a_3, \ldots, a_n$ are all even. If $n = 2k + 1$, then $k + 1$ even numbers are in that sequence. but there are only k even numbers between 1 and n, thus one of them has to be odd, and the product become even.

3. 2 non-equal points on a sphere determine a great circle. Then use pigeonhole principle. There are 3 points and 2 hemispheres that determined by the great circle. One has to have 2 of them. Together with the points on the great circle, 4 points have to lie in a hemisphere.

Problem 4

There are n^2 pairs of (j, k), since it is run n times , there are $(n^2)^n$ possible operations. There can be $n!$ outcomes. $\frac{(n^2)^n}{n!}$ is not a integer, since n is relatively prime to $n-1$ for $n > 2$. Thus this is not a k-to-1 mapping for some integer k. Then some permutation will show up more often.

Problem 5

procedure genPerm(n)
    $A \leftarrow [0, 1, 2, \ldots, n-1]$
    while $n > 1$
        $n \leftarrow n - 1$
        $j \leftarrow$ random number in the set $\{0, \ldots, n\}$
        SWAP($A[j]$, $A[n]$)
    return A

There are only $n!$ possible procedures, since each $A[j]$ have j possible position to swap, which also prove each permutation uniquely determined by each procedure.

Problem 6

Any 3 professor can open every lock, implies each lock have at least 3 keys. If that's not true, then it is possible to chose 3 professor, where non of them have the key to the lock.
Any 2 professor have a lock they can't open, and the above condition made each lock they can't open must be distinct. We have $\binom{5}{2} = 10$ locks as the lower bound of locks. We can produce a table of lock vs professors. Where 1 means there is a key for the professor to a particular lock, 0 means other wise. There are only $\binom{5}{2}$ ways to have a string of 3 ones and 2 zeros. Thus we can generate the following table effortlessly.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	0	0	0	0
2	1	1	1	0	0	0	1	1	1	0
3	1	0	0	1	1	0	1	1	0	1
4	0	1	0	1	0	1	1	0	1	1
5	0	0	1	0	1	1	0	1	1	1

It in fact works. Thus 10 are the minimal number of locks.

Problem 7

Let $P(n)$ be the predicate "n lines divide the a paper into regions, and it can be colored into blue and red such that no region share the same edge have the same color."
Base case: $P(n)$ is true.
Induction step: Suppose $P(n)$ is true, then $P(n+1)$ is true.
Use n lines to create a $P(n)$. Then add the $n+1$th line, let one side of the line's color be the same, let the other side of the line's color be reversed. The result is $P(n+1)$.
Thus $P(n)$ is true for all n.

Problem 8

It was shown in class, for n ball n bin problem, we expect $\frac{n}{e}$ bins have exactly 1 ball when n is large. Let $E(n)$ be the expected amount of throws. We have the relation $E(n) = E(n - \frac{n}{e}) + 1$ or $E(n) = E(n(1 - \frac{1}{e})) + 1$. let $1 - \frac{1}{e} = \frac{1}{b}$, then $E(n) = E(\frac{n}{b}) + 1$. With master theorem, we have $\Theta(E(n)) = \Theta(\log n)$.